Remarks on concavity of the auxiliary function appearing in quantum reliability function

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Abstract — Concavity of the auxiliary function which appears in the random coding exponent as the lower bound of the quantum reliability function for general quantum states is proved for 0 ≤ s ≤ 1 in two dimensional case.

I. INTRODUCTION

The reliability function of classical-quantum channel is defined by

\[ E(R) \equiv - \lim \inf_{n \to \infty} \frac{1}{n} \log P_e(2^nR, n), \quad 0 < R < C, \quad (1) \]

where C is a classical-quantum capacity, R is a transmission rate \( R \equiv \frac{\log_2 M}{n} \) (n and M represent the number of the code words and the messages, respectively), \( P_e(M, n) \) can be taken any minimal error probabilities of \( \min_{W, X} P(W, X) \) or \( \min_{W, X} P_{\text{max}}(W, X) \). Let \( S_w \) be the density operator corresponding to the code word \( w \) chosen from the code (block) \( \mathcal{W} = \{ w^1, w^2, \ldots, w^M \} \). For details, see [2, 3].

The lower bound for the quantum reliability function defined in Eq.(1), when we use random coding, is given by

\[ E(R) \geq E_q^2(R) \equiv \max_n \sup_{0 < s \leq 1} \left[ E_q(\pi, s) - sR \right], \]

where \( \pi = \{ \pi_1, \pi_2, \ldots, \pi_a \} \) is a priori probability distribution satisfying \( \sum_{i=1}^a \pi_i = 1 \) and

\[ E_q(\pi, s) = - \log G(s), \]

\[ G(s) = \text{Tr} \left[ A(s)^{1+s} \right], \]

\[ A(s) = \sum_{i=1}^a \pi_i S_i^{1+s}, \]

where each \( S_i \) is density operator which corresponds to the output state of the classical-quantum channel \( i \rightarrow S_i \) from the set of the input alphabet \( A = \{ 1, 2, \ldots, a \} \) to the set of the output quantum states in the Hilbert space \( \mathcal{H} \).

II. A SUFFICIENT CONDITION ON CONCAVITY OF THE AUXILIARY FUNCTION

Proposition 1 [1] For any real number \( s \) \( (-1 < s \leq 1) \), density operators \( S_i \) \( (i = 1, \ldots, a) \) and a priori probability \( \pi = \{ \pi_i \} \) \( (i = 1, \ldots, a) \) such that \( A(s) \) is invertible, if the trace inequality

\[ \text{Tr} \left[ A(s)^s \sum_{i=1}^a \pi_i S_i^{1+s} \log S_i^{1+s} \right]^2 \]

\[ \geq \text{Tr} \left[ A(s)^{-1+s} \left( \sum_{i=1}^a \pi_i H(S_i^{1+s}) \right)^2 \right] \geq 0 \quad (2) \]

holds, then the auxiliary function

\[ E_q(\pi, s) = - \log \left[ \text{Tr} \left\{ \left( \sum_{i=1}^a \pi_i S_i^{1+s} \right)^{1+s} \right\} \right] \]

is concave in \( s \). Where \( H(x) = -x \log x \) is the operator entropy.

III. SPECIAL CASES

Theorem 1 Eq. (2) can be proved in some special cases.

(i) The case \( a = 2, s = 1 \).

(ii) The case \( a = 2, s = 0 \).

Remark 1 (i) holds for any \( a \) and \( \pi \). (ii) holds for any \( \pi \) but not known for any \( a \).

Lemma 1 If the following conditions

(i) \( t_1 a_1 + t_2 a_2 \geq b_1 + b_2 \)

(ii) \( a_1 + a_2 \geq t_1^{-1} b_1 + t_2^{-1} b_2 \)

hold for \( t_1, t_2, a_1, a_2, b_1, b_2 > 0 \), then for any \( s \) \( (0 \leq s \leq 1) \), we have

\[ t_1^{-s} a_1 + t_2^{-s} a_2 \geq t_1^{-1+s} t_1^{-1+s} b_1 + t_2^{-1+s} b_2. \]

Since we have the condition (i) and (ii) in the Lemma 1 from the Theorem 1, we have the following theorem.

Theorem 2 Let \( A, B \) be the two dimensional positive operators. Then for any \( s \) \((0 \leq s \leq 1)\), we have

\[ \text{Tr}[(A + B)^s (A \log A)^{\frac{s}{2}} + B \log B)^{\frac{s}{2}}] \]

\[ -\text{Tr}[(A + B)^{-1+s} (A \log A + B \log B)^{\frac{s}{2}}] \geq 0. \]

Remark 2 Theorem 2 holds for any \( \pi_1 \) and \( \pi_2 \).

REFERENCES

