ON A THEOREM CONCERNING TO UNIVALENT FUNCTIONS LINEARLY ACCESSIBLE BY M. BIENACKI

(ENGLISH TRANSLATION OF "SUR UN THÉORÈME CONCERNANT LES FONCTIONS UNIVALENTES LINÉAIREMENT ACCESSIBLES DE M. BIENACKI" BY A. BIELECKI AND Z. LEWANDOWSKI)

1. A complex function

$$f(z) = a_1 z + a_2 z^2 + \cdots, \qquad a_1 \neq 0, \tag{1}$$

holomorphic in the unit disk $C_1 = \{z : |z| < 1\}$, is called *close-to-convex* if there exists a function $g(z) = b_1 z + b_2 z^2 + \cdots$, $b_1 \neq 0$, univalent and convex in C_1 such that

$$\operatorname{Re} \frac{f'(z)}{g'(z)} > 0, \qquad |z| < 1. \tag{2}$$

This conditions are fulfilled, then the function f must be univalent in $\mathbb{D}$ and

$$\operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} > 0, \qquad |z| < 1. \tag{3}$$

Close-to-convex functions were introduced in the theory of univalent functions by Kaplan in 1952 [3, p.169]. Z. Lewandowski noted that the notion of close-to-convex functions was defined already in 1936 by M. Bienacki [2] as a linearly accessible function. A function f of the form (1) is called linearly accessible if the complementary set of $f(C_1)$ in the plane can be covered by the closed half-lines do not intersect in pairs, which means that a point which belongs to two distinct half-lines must be the end point of at least one of them.

However, Lewandowski has shown that any linearly accessible functions are close-to-convex [4], and conversely, every close-to-convex functions are linearly accessible [5]. The proof of this last theorem, given in [5], is quite long and painful. The purpose of this note is to give another demonstration shorter, based on a simple principle that we used in [1].

2. Suppose that f is a function of the form (1) close-to-convex in $\mathbb{D}$ and set

$$F(z,t) = f(z) + tzg'(z)$$

for $z \in \mathbb{D}$ and $t \in [0,\infty)$. Since

$$\frac{\partial_z F(z,t)}{g'(z)} = \frac{z\partial_z F(z,t)}{\partial_t F(z,t)} = \frac{f'(z)}{g'(z)} + t\left[1 + \frac{zg''(z)}{g'(z)}\right],$$

it follows from (2) and (3) that the function $F(z,t)$ is close-to-convex univalent in $\mathbb{D}$ for any fixed $t \in [0,\infty)$, and in addition, we have $\operatorname{Re} z\partial_z F(z,t)/\partial_t F(z,t) > 0$ for all $z \in \mathbb{D}$ and $t \in [0,\infty)$[1], the following property of the function F is obtained (cf. [1, p.47]):

Property 1. *If $\rho \in (0,1)$ and $t_1 \leq t_2$, and if $C_\rho = \{z : |z| < \rho\}$, then the domain $F(\overline{C_\rho}, t_1) = \{\zeta : \zeta = F(z,t_1), |z| \leq \rho\}$ is contained in the domain $F(C_\rho, t_2)$.*

3. Fix an integer $n \geq 2$ and assume that

$$r = 1 - \frac{1}{n}, \qquad f_n(z) = f(rz). \tag{4}$$

[1]This condition is interpreted as follows: When the papramete t increases, the boundary curve of $f(C_\rho, t), \rho \in (0,1)$, moves so that the direction of the instantaneous velocity of any points P of the curve forms an acute angle with the conducted outside the normal curve (vector) at point P. Thus widening the domain $F(C_\rho, t)$ as t increases.

It is seen that for $t \geq 0$

$$\Gamma(t) = \{\zeta : \zeta = F(re^{i\theta}, t), \theta \in [0, 2\pi)\}$$

is a simple curve which is the boundary of the domain $F(C_r, t)$, while all

$$\ell(\theta) = \{\zeta : \zeta = F(re^{i\theta}, t),\ t \geq 0\},\ \ \theta : \text{fixed}$$

is a closed half-line whose endpoint is $f(re^{i\theta}) = f_n(e^{i\theta})$ is located on the boundary curve $\Gamma(0)$ of the domain $f_n(C_1)$.

Suppose $0 \leq |\theta - \sigma| < 2\pi$ and the half-lines $\ell(\theta)$ and $\ell(\sigma)$ have a common point

$$\zeta = F(re^{i\theta}, t) = F(re^{i\sigma}, s).$$

The function $F(z, t)$ is univalent for any fixed $t \geq 0$ and $t \neq s$. But this leads to a contradiction, because in this case one curve $\Gamma(t)$ or $\Gamma(s)$ must be contained in the interior of the other, under Property 1. We have shown that for $\theta \in [0, 2\pi)$, the half-lines $\ell(\theta)$ are disjoint in pairs. Note also none of the rays $\ell(\theta)$ may pass through the points contained in the interior of the curve $\Gamma(0)$, if a ray $\ell(\theta)$ should cut the curve at a point $f(re^{i\tau})$, where $\tau \neq \theta$, which would be the endpoint of another half-line $\ell(\tau)$.

4. Now fix a point ζ located outside the circle $\Gamma(0)$, and denote by $\psi(\theta)$ the angle between the real axis and the ray $m(\theta)$ from the point $f(re^{i\theta})$ and passing through the point ζ, and $\varphi(\theta)$ is the angle between the real axis and the ray $\ell(\theta)$. However, under the definition of $\ell(\theta)$, $\varphi(\theta) = \arg \partial_t F(re^{i\theta}) = \arg\{re^{i\theta} g'(re^{i\theta})\}$.

But the function g is convex and therefore increases with the angle $\varphi(\theta)$ and θ and $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$. On the other hand, because of $\psi(\theta + 2\pi) = \psi(\theta)$, ζ is located on the contour (boundary?) $\Gamma(0)$. So the increase of the angle $\chi = \varphi(\theta) - \psi(\theta)$ corresponding to the increase 2π of the parameter θ is equal to 2π, and therefore, there is a real number θ_0 and an integer k such as $\chi(\theta_0) = 2\pi k$; but this means that the ray $\ell(\theta_0) = m(\theta_0)$ passes through the point ζ.

We have demonstrated the lines $\ell(\theta)$, where $\theta \in [0, 2\pi)$ cover all the points outside the contour $\Gamma(0)$ limiting the domain $f_n(C_1) = f(C_r)$. Points belonging to the same time range are the origins of rays $\ell(\theta)$ and in Section 3 we found that these rays do not intersect each other. So the function $f_n(z)$ is linearly accessible from the definition due to Bienacki, and this result is obviously true for $n = 2, 3, \cdots$.

According to (4), the sequence of functions $\{f_n\}$ converges uniformly to the function f in any circle C_ρ, where $\rho \in (0, 1)$, and these functions are linearly accessible. Under a theorem of Bienacki that is enough for the limit function f is also linearly accessible, which completes our proof.

References

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Annales Polonici Mathematici 12 (1962), 61–63