AHLFORS'S QUASICONFORMAL EXTENSION CONDITION AND Φ-LIKENESS

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Abstract. The notion of Φ-like functions is known to be a necessary and sufficient condition for univalence. By applying the idea, we derive several necessary conditions and sufficient conditions for that an analytic function defined on the unit disk is not only univalent but also has a quasiconformal extension to the Riemann sphere, as generalizations of well-known univalence and quasiconformal extension criteria, in particular, Ahlfors's quasiconformal extension condition.

1. Introduction

Let $\mathbb{C}$ be the complex plane, $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ for $r > 0$ and $\mathbb{D} := \mathbb{D}_1$. We denote by $\mathcal{A}$ the family of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic on $\mathbb{D}$ and $\mathcal{S}$ the subclass of $\mathcal{A}$ whose members are univalent, that is, one-to-one on $\mathbb{D}$. For standard terminology in the theory of univalent functions, see for instance [16] and [9]. Let $k \in [0, 1)$ be a constant. Then a homeomorphism f of $G \subset \mathbb{C}$ is said to be *k-quasiconformal* if $\partial_z f$ and $\partial_{\bar{z}} f$ in the distributional sense are locally integrable on G and fulfill $|\partial_{\bar{z}} f| \leq k|\partial_z f|$ almost everywhere in G. If we do not need to specify k, we will simply call that f is *quasiconformal*.

We begin our argument by observing a fundamental composition property of functions. Let f, $g \in \mathcal{A}$ and Q be an analytic function defined on $f(\mathbb{D})$ which satisfies $g = Q \circ f$. Then a necessary and sufficient condition for univalence of g on $\mathbb{D}$ is that f and Q are univalent on their respective domains. Let us apply this fact to derive a univalence criterion. In order to demonstrate it, we set one example with a condition for *λ-spirallike functions*, i.e., $\mathrm{Re}\,\{e^{-i\lambda} zg'(z)/g(z)\} > 0$ is satisfied for all $z \in \mathbb{D}$, where $-\pi/2 < \lambda < \pi/2$ (see e.g. [8, p.52]). Note that all λ-spirallike functions are univalent on $\mathbb{D}$. In view of the relationship $g = Q \circ f$, the condition of λ-spirallikeness of g is equivalent to

$$\mathrm{Re}\, \frac{zf'(z)}{\Phi(f(z))} > 0, \tag{1}$$

where $\Phi(w) = e^{i\lambda} Q(w)/Q'(w)$. We conclude that (1) is a sufficient condition of univalence of f. This is an essential idea of the notion of "Φ-like functions".

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Definition. A function $f \in \mathcal{A}$ is said to be Φ-*like* if there exists an analytic function Φ defined on $f(\mathbb{D})$ such that (1) holds for all $z \in \mathbb{D}$.

Remark 1.1. The inequality (1) implies $\Phi(0) = 0$ and $\operatorname{Re}\Phi'(0) > 0$.

The notion of Φ-likeness also turns out a necessary condition for univalence of f. In fact, if f is univalent in $\mathbb{D}$ then we can define Φ by means of $Q := g \circ f^{-1}$, where g is a spirallike function. Consequently, we obtain the following:

Theorem A. *A function $f \in \mathcal{A}$ is univalent in $\mathbb{D}$ if and only if f is Φ-like.*

Remark 1.2. If we choose $\Phi(w) = e^{i\lambda}w$ then it immediately follows the condition for λ-spirallikeness.

The concept of Φ-like function was introduced by Kas'yanyuk [12] and Brickman [7] independently. The reader is referred to [3, §7] which contains some more information about Φ-like functions. The above instructive characterization of Φ-like functions is due to Ruscheweyh [17]. Furthermore, he gave the following two generalizations of well-known univalence conditions by the same technique as it.

Theorem B (Generalized Becker condition [17])**.** *Let $f \in \mathcal{A}$. Then f is univalent if and only if there exists an analytic function Ω on $f(\mathbb{D})$ such that*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)} + zf'(z)\Omega(f(z))\right| \leq 1 \tag{2}$$

for all $z \in \mathbb{D}$.

Theorem C (Generalized Bazilevič functions [17])**.** *Let $f \in \mathcal{A}$, $p(z)$ with $p(0) = p'(0) - 1 = 0$ be starlike univalent in $\mathbb{D}$ and $s = \alpha + i\beta \in \mathbb{C}$, $\operatorname{Re} s > 0$. Then f is univalent in $\mathbb{D}$ if and only if there exists an analytic function $\Psi(w)$ on $f(\mathbb{D})$ with $\Psi(0) \neq 0$ such that*

$$\operatorname{Re}\frac{f'(z)\,(f(z)/z)^{s-1}}{(p(z)/z)^{\alpha}}\Psi(f(z)) > 0 \tag{3}$$

for all $z \in \mathbb{D}$.

Remark 1.3. The choices $\Omega \equiv 0$ and $\Psi \equiv e^{i\alpha}$ correspond to the original univalence conditions due to Becker [5] and Bazilevič [4] respectively.

Since Ref. [17] is not published, we outline proofs of Theorem B and Theorem C for convenience. We can show the case (2) from the fact that $g(z) = Q(f(z))$ with $Q''/Q' = \Omega$ satisfies original Becker's univalence condition, and the case (3) from that the function $g(z) = Q(f(z))$ with $Q(w) = \left(s\int_0^w t^{s-1}\Psi(t)dt\right)^{1/s} = \Psi(0)w + \cdots$ is Bazilevič and hence univalent in $\mathbb{D}$. The other directions of Theorem B and Theorem C can be easily proved to define Ω and Ψ by $Q(w) = g(f^{-1}(w))$, $w \in f(\mathbb{D})$, where g is a suitable function which satisfies Becker's condition or the Bazilevič function, respectively.

The main aim of this paper is to derive several necessary conditions and sufficient conditions for that a function $f \in \mathcal{A}$ is univalent in $\mathbb{D}$ and extendible to a quasiconformal mapping to the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ as an application of Ruscheweyh's characterization of Φ-like functions. These results are based on well-known univalence and quasiconformal extension criteria, for instance, the following Ahlfors's quasiconformal extension condition. Here $\mathcal{S}(k)$ $(0 \le k < 1)$ is the family of functions which are in $\mathcal{S}$ and can be extended to k-quasiconformal mappings to $\widehat{\mathbb{C}}$. The subfamily of $\mathcal{S}(k)$ whose k-quasiconformal extensions fix ∞ is denoted by $\mathcal{S}_0(k)$.

Theorem D ([1]). *Let $f \in \mathcal{A}$. If there exists a k, $0 \le k < 1$, such that for a constant $c \in \mathbb{C}$ and all $z \in \mathbb{D}$*

$$\left| c|z|^2 + (1-|z|^2)\frac{zf''(z)}{f'(z)} \right| \le k, \tag{4}$$

then $f \in \mathcal{S}_0(k)$ (the condition $|c| \le k$ is included in (4) [11, Remark 1.1 and 1.2]).

The following is our main theorem:

Theorem 1. *Let $f \in \mathcal{A}$ and $k \in [0,1)$. If there exists a $k' \in [0,k)$ and a conformal mapping Q with $Q'(0) \neq 0$ defined on $f(\mathbb{D})$, has a $(k-k')/(1-kk')$-quasiconformal extension to $\widehat{\mathbb{C}}$ and satisfies for a constant $c \in \mathbb{C}$ and for all $z \in \mathbb{D}$*

$$\left| c|z|^2 + (1-|z|^2)\left\{ \frac{zf''(z)}{f'(z)} + zf'(z)\Omega(f(z)) \right\} \right| \le k', \tag{5}$$

where $\Omega = Q''/Q'$, then $f \in \mathcal{S}(k)$. Conversely, if $f \in \mathcal{S}(k)$ then there exists a $k' \in [0,1)$ and a conformal mapping Q with $Q'(0) \neq 0$ defined on $f(\mathbb{D})$, has a $(k+k')/(1+kk')$-quasiconformal extension to $\widehat{\mathbb{C}}$ such that the inequality (5) holds for a constant $c \in \mathbb{C}$ and for all $z \in \mathbb{D}$.

Remark 1.4. The case $Q(z) = z$ is Theorem D. Further, if the extended quasiconformal mapping of Q does not take the value ∞ in $\mathbb{C}$, then $\mathcal{S}(k)$ can be replaced by $\mathcal{S}_0(k)$.

In contrast to the case of univalent functions, it is not always true that if $g \circ f$ has a quasiconformal extension then so do f and g as well. This is the reason why the function Q is required some bothersome assumptions in Theorem 1. On the other hand, if we give a specific form of Q, then it can be obtained several new quasiconformal extension criteria which are of practical use. We will discuss it in the last section.

2. Preliminaries

Let $f_t(z) = f(z,t) = \sum_{n=1}^{\infty} a_n(t) z^n$, $a_1(t) \neq 0$, be a function defined on $\mathbb{D} \times [0,\infty)$ and analytic in $\mathbb{D}$ for each $t \in [0,\infty)$, where $a_1(t)$ is a locally absolutely continuous function on $[0,\infty)$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. f_t is said to be a *Löwner chain* if f_t is univalent on $\mathbb{D}$ for each $t \in [0,\infty)$ and satisfies $f_s(\mathbb{D}) \subsetneq f_t(\mathbb{D})$ for $0 \le s < t < \infty$.

The following necessary and sufficient condition for Löwner chains due to Pommerenke is well-known.

Theorem E ([15]). *Let $0 < r_0 \le 1$. Let $f(z,t)$ be a function defined above. Then the function $f(z,t)$ is a Löwner chain if and only if the following conditions are satisfied:*

1. *The function $f(z,t)$ is analytic in $\mathbb{D}_{r_0}$ for each $t \in [0,\infty)$, locally absolutely continuous in $[0,\infty)$ for each $z \in \mathbb{D}_{r_0}$ and*

$$|f(z,t)| \le K|a_1(t)| \qquad (z \in \mathbb{D}_{r_0},\ \text{a.e. } t \in [0,\infty))$$

for some positive constants K.

2. *There exists a function $p(z,t)$ analytic in $\mathbb{D}$ for each $t \in [0,\infty)$ and measurable in $[0,\infty)$ for each $z \in \mathbb{D}$ satisfying*
$$\operatorname{Re} p(z,t) > 0 \qquad (z \in \mathbb{D},\ t \in [0,\infty))$$
such that
$$\partial_t f(z,t) = z\partial_z f(z,t) p(z,t) \qquad (z \in \mathbb{D}_{r_0},\ \text{a.e. } t \in [0,\infty)).$$

Remark 2.1. In usual formulation, a Löwner chain $f(z,t)$ does not have any constant terms. On the other hand, a function $f^*(z,t) := f(z,t) + c$ satisfies all the conditions of the definition of Löwner chains and the sufficient conditions of Theorem E with a modification of K, where c is a complex constant which does not depend on t and z. For this reason here and hereafter we also admit such f^* as Löwner chains.

The next theorem which is due to Becker plays a central role in our argument.

Theorem F ([5, 6]). *Suppose that $f_t(z) = f(z,t)$ is a Löwner chain for which $p(z,t)$ in the part 2 of Theorem E satisfies the condition*
$$\begin{aligned} p(z,t) \in U(k) &:= \left\{ w \in \mathbb{C} : \left| \frac{w-1}{w+1} \right| \leq k \right\} \\ &= \left\{ w \in \mathbb{C} : \left| w - \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2} \right\} \end{aligned}$$
for all $z \in \mathbb{D}$ and almost all $t \in [0,\infty)$. Then $f(z,t)$ admits a continuous extension to $\overline{\mathbb{D}}$ for each $t \geq 0$ and the map $\hat{f}$ defined by
$$\hat{f}(re^{i\theta}) = \begin{cases} f(re^{i\theta}, 0), & \text{if} \quad r < 1, \\ f(e^{i\theta}, \log r), & \text{if} \quad r \geq 1, \end{cases}$$
is a k-quasiconformal extension of f_0 to $\mathbb{C}$. In other words, if f_t is normalized by $f_t(0) = 0$ then the above theorem gives a sufficient condition for $f_0 \in \mathcal{S}_0(k)$.

To discuss a problem of quasiconformal extensions, we need several background of the theory of quasiconformal mappings. We list a few fundamental properties of the theory which will be used later. For more detail, readers are recommend [2] or [14] for general reference of the theory.

Lemma G. *Fundamental properties of quasiconformal mappings ($k, k_1, k_2 \in [0,1)$).*

1. *A composition of k_1-quasiconformal mapping and k_2-quasiconformal mapping is a $(k_1+k_2)/(1+k_1k_2)$-quasiconformal mapping.*
2. *If f is a k-quasiconformal mapping, then f^{-1} is also a k-quasiconformal mapping.*
3. *A 0-quasiconformal mapping is conformal.*

3. Proof of Theorem 1

Firstly we show the first part of Theorem 1. Define
$$F(z,t) := Q(f(e^{-t}z)) + (1+c)^{-1}(e^t - e^{-t})zQ'(f(e^{-t}z))f'(e^{-t}z). \tag{6}$$

We take into account that $1 + c \neq 0$ since the inequality (5) implies $|c| \leq k' < 1$ (see [11, Remark 1.1 and 1.2]). Then we have

$$\begin{aligned} &\left|\frac{\partial_t F(z,t) - z\partial_z F(z,t)}{\partial_t F(z,t) + z\partial_z F(z,t)}\right| \\ &\quad = \left| e^{-2t}c + (1-e^{-2t})\left\{ e^{-t}z\frac{f''(e^{-t}z)}{f'(e^{-t}z)} + e^{-t}zf'(e^{-t}z)\Omega(f(e^{-t}z))\right\}\right| \end{aligned} \tag{7}$$

where $\Omega = Q''/Q'$. The right-hand side of (7) is always less than or equal to k' from (5). Hence the mapping $g := Q \circ f$ has a k'-quasiconformal extension to $\mathbb{C}$ by Theorem E and Theorem F. Since Q can extend a $(k-k')/(1-kk')$-quasiconformal mapping to $\widehat{\mathbb{C}}$, we conclude by Theorem G that $f = Q^{-1} \circ g \in \mathcal{S}(k)$.

The second part of Theorem 1 is verified to give a specific Q. Let $h \in \mathcal{A}$ which satisfying

$$\left| c_0|z|^2 + (1-|z|^2)\frac{zh''(z)}{h'(z)} \right| \leq k' \tag{8}$$

for all $z \in \mathbb{D}$ with a certain $c_0 \in \mathbb{C}$. Then we have $f \in \mathcal{S}(k)$ by Theorem. Using such h, we set $Q := h \circ f^{-1}$. Q is conformal on $f(\mathbb{D})$ and $Q'(0) = h'(0)/f'(0) \neq 0$. Since $f \in \mathcal{S}(k)$, Q admits a $(k_1+k_2)/(1+k_1k_2)$-quasiconformal extension to $\mathbb{C}$ by Theorem G. Further, $Q \circ f = h$ satisfies the inequality (8) which is equivalent to (5). Consequently, the function Q is our desired one. □

4. Further results

It is possible to derive other similar necessary conditions and sufficient conditions for quasiconformal extensions as Theorem 1. We select one example out of a large variety of possibilities.

Theorem 2. *Let $f \in \mathcal{A}$ and $k \in [0,1)$. If there exists a $k' \in [0,k)$ and a conformal mapping Q with $Q'(0) \neq 0$ defined on $f(\mathbb{D})$, has a $(k-k')/(1-kk')$-quasiconformal extension to $\widehat{\mathbb{C}}$ such that*

$$f'(z)Q'(f(z)) \in U(k') \tag{9}$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{S}(k)$, where $U(k')$ is the disk defined in Theorem F. Conversely, if $f \in \mathcal{S}(k)$ then there exists a $k' \in [0,1)$ and a conformal mapping Q with $Q'(0) \neq 0$ defined on $f(\mathbb{D})$, has a $(k+k')/(1+kk')$-quasiconformal extension to $\widehat{\mathbb{C}}$ such that the inequality (9) holds for all $z \in \mathbb{D}$.

In the proof of Theorem 2, the following quasiconformal extension criterion based on the Noshiro-Warschawski theorem is used:

Theorem H ([18, 13, 10])**.** *Let $f \in \mathcal{A}$ and $k \in [0,1)$. If f satisfies $f'(z) \in U(k)$ for all $z \in \mathbb{D}$, then $f \in \mathcal{S}_0(k)$.*

Proof of Theorem 2. The proof follows the similar lines as the proof of Theorem 1. Let us firstly put

$$F(z,t) := Q(f(z)) + (e^t - 1)z. \tag{10}$$

Then calculations show that
$$\frac{z\partial_z F(z,t)}{\partial_t F(z,t)} = \frac{1}{e^t}\,Q'(f(z))f'(z) + \left(1 - \frac{1}{e^t}\right).$$
Thus $g := Q\circ f \in \mathcal{S}(k')$ by Theorem E and Theorem F. Since Q has a $(k-k')/(1-kk')$-quasiconformal extension to $\widehat{\mathbb{C}}$ by the assumption, it follows from Theorem G that $f = g\circ Q^{-1} \in \mathcal{S}(k)$.

Conversely, suppose that $f \in \mathcal{S}(k)$. Then following the line of the proof of Theorem 1, one can deduce that $Q := h\circ f^{-1}$ is the function which satisfies the assertion of the theorem, where h is a function in $\mathcal{A}$ satisfying $f'(z) \in U(k')$ for all $z \in \mathbb{D}$ (This implies $h \in \mathcal{S}(k')$ by Theorem H).

□

Various similar results as Theorem 2 can be proved to choose the other univalence criterion and set a suitable Löwner chain. For example, the condition
$$\frac{zf'(z)}{\Phi(f(z))} \in U(k')$$
which is based on the definition of Φ-like functions is given by the Löwner chain
$$F(z,t) = e^t Q(f(z)),$$
or
$$\frac{f'(z)\,(f(z)/z)^{s-1}}{(p(z)/z)^{\alpha}}\Psi(f(z)) \in U(k')$$
which is based on the definition of the Bazilevič functions is given by
$$F(z,t) = \left\{Q(f(z))^s + s(e^t-1)p(z)^{\alpha}z^{i\beta}\right\}^{1/s},$$
where Φ and Ψ are functions defined in Section 1.

5. Applications

In this section we consider several applications of theorems we have obtained in previous sections, in particular, Theorem 1 and Theorem 2. Two specific forms of the function Q which is univalent on a certain domain and can be extended to a quasiconformal mapping to $\widehat{\mathbb{C}}$ are given.

We remark that in the cases below Q does not need to be normalized by $Q(0) = 0$. For Löwner chains $F(z,t)$ defined in (6) and (10) we have $F(0,t) = Q(0)$ which implies that in both cases a constant term of $F(z,t)$ does not depend on t. Hence, as we noted in Remark 2.1, $F(z,t)$ is a Löwner chain even though $Q(0) \neq 0$. This fact allows us to avoid some technical complications.

5.1. **Möbius transformations.**

Let Q_1 be the Möbius transformation given by
$$Q_1(w) := \frac{\alpha w + \beta}{\gamma w + \delta} \quad (\alpha,\,\beta,\,\gamma,\,\delta \in \mathbb{C},\ \gamma \neq 0,\ \alpha\delta - \beta\gamma = 1). \tag{11}$$
For a given $f \in \mathcal{A}$, Q_1 with $-\delta/\gamma \notin f(\mathbb{D})$ is considered as a function conformal on $f(\mathbb{D})$ and has a 0-quasiconformal extension to $\widehat{\mathbb{C}}$. We note that Q_1 is the unique function

which it can be chosen as Q in Theorem 1 and Theorem 2 without any restrictions on the shape of $f(\mathbb{D})$.

Simple calculations show that $Q_1'(w) = 1/(\gamma w + \delta)^2$ and $Q_1''(w)/Q_1'(w) = -2/(w + (\delta/\gamma))$. Hence by defining $Q := Q_1$ we obtain the following new quasiconformal extension criteria as corollaries of Theorem 1 and Theorem 2.

Corollary 3. *Let $f \in \mathcal{A}$ and $k \in [0, 1)$. If f satisfies*

$$\left| c_1|z|^2 + (1 - |z|^2) \left\{ \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z) - c_2} \right\} \right| \leq k$$

for some constants c_1, $c_2 \in \mathbb{C}$ and for all $z \in \mathbb{D}$, then $f \in \mathcal{S}(k)$.

Corollary 4. *Let $f \in \mathcal{A}$ and $k \in [0, 1)$. If f satisfies*

$$\frac{f'(z)}{(\gamma f(z) + \delta)^2} \in U(k)$$

for some constants γ, $\delta \in \mathbb{C}$, $\gamma \neq 0$, and for all $z \in \mathbb{D}$, then $f \in \mathcal{S}(k)$.

Remark 5.1. We assumed that $\gamma \neq 0$ in (11) because in the case when $\gamma = 0$ the function Q_1 is an affine transformation and thus Corollary 3 and Corollary 4 are nothing but Theorem D and Theorem H, respectively.

Remark 5.2. In the above corollaries $\mathcal{S}(k)$ cannot be replaced by $\mathcal{S}_0(k)$, because $\gamma \neq 0$ which implies Q_1 does not fix ∞.

5.2. **Sector domain.** We may set the function Q under the assumption that the image of $\mathbb{D}$ under $f \in \mathcal{A}$ is contained in a quasidisk which has a special shape. For instance, we suppose that $f(\mathbb{D})$ lies in the sector domain

$$\Delta(w_0, \lambda_0, a) := \{w \in \mathbb{C} : \pi\lambda_0 < \arg(w - w_0) < \pi(a + \lambda_0)\}$$

for $w_0 \in \mathbb{C}\backslash f(\mathbb{D})$, $\lambda_0 \in [0, 2)$ and $a \in (0, 2)$. Then, we define Q in Theorem 1 and Theorem 2 by Q_2,

$$Q_2(w) := \left(e^{-i\pi\lambda_0}(w - w_0)\right)^{1/a},$$

which maps $\Delta(w_0, \lambda_0, a)$ conformally onto the upper half-plane. It is verified that Q_2 can be extended to a $|1 - a|$-quasiconformal automorphism of $\mathbb{C}$ as follows: Let us set

$$P_1(z) := z^{1/(2-a)}, \quad P_2(z) := |z|^{(2-a)/a} \frac{z}{|z|},$$

respectively. Then the function P defined by

$$P(z) := \begin{cases} z^{1/a}, & \textit{if} \quad z \in \Delta(0, 0, a), \\ -(P_2 \circ P_1)(e^{-i\pi a} z), & \textit{if} \quad z \in \overline{\Delta(0, a, 2 - a)}, \end{cases}$$

is a $|1 - a|$-quasiconformal automorphism of $\mathbb{C}$. After composing proper affine transformations we obtain the desired extension of Q_2.

Since $Q_2''(w)/Q_2'(w) = ((1/a) - 1)/(w - w_0)$, we deduce the following:

Corollary 5. *Let $f \in \mathcal{A}$ and $k \in [0,1)$. We assume that $f(\mathbb{D})$ is contained in the sector domain $\Delta(w_0, \lambda_0, a)$. If f satisfies*

$$\left| c|z|^2 + (1-|z|^2)\left\{ \frac{zf''(z)}{f'(z)} + \left(\frac{1}{a} - 1\right) \frac{zf'(z)}{f(z) - w_0} \right\} \right| \le k$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{S}_0(\ell)$, where $\ell = (k + |1-a|)/(1 + k|1-a|)$.

To state the next corollary we shall set $Q_3(w) := Q_2(w)/Q_2'(0)$, so that $Q_3'(w) = (1 - (w/w_0))^{(1/a)-1}$ and hence $f'(0)Q_3'(0) = 1 \in U(k)$ for any $k \in [0,1)$.

Corollary 6. *Let $f \in \mathcal{A}$ and $k \in [0,1)$. We assume that $f(\mathbb{D})$ is contained in the sector domain $\Delta(w_0, \lambda_0, a)$. If f satisfies*

$$f'(z)\left(1 - \frac{f(z)}{w_0}\right)^{(1/a)-1} \in U(k)$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{S}_0(\ell)$, where $\ell = (k + |1-a|)/(1 + k|1-a|)$.

As a special case of Corollary 6, if we can choose $w_0 = -1$ and $a = 1/2$, then we will have a $(2k+1)/(k+2)$-quasiconformal extension criterion $f'(z)(1 + f(z)) \in U(k)$.

Corollary 5 and Corollary 6 may seem to be less useful in practical situations because of their assumption that $f(\mathbb{D})$ lies in a sector domain. It will, however, be used effectively sometimes for the following reason. When we investigate whether $f \in \mathcal{S}$ is contained in $\mathcal{S}_0(k)$ or not, it is enough to deal with only bounded components of $\mathcal{S}$, because otherwise $f \notin \mathcal{S}_0(k)$ for any $k \in [0,1)$. In addition, if f is bounded, then there always exists a sector domain which contains $f(\mathbb{D})$.

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