ON STRONGLY STARLIKE AND CONVEX FUNCTIONS OF ORDER α AND TYPE β

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Abstract. In this note we investigate the inclusion relationship between the class of strongly starlike functions of order α and type β, $\alpha \in (0,1]$ and $\beta \in [0,1)$, which satisfy

$$\left|\arg\left\{\frac{zf'(z)}{f(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha$$

and the class of strongly convex functions of order α and type β which satisfy

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha$$

in the unit disk, where f is an analytic function defined on the unit disk and satisfies $f(0) = f'(0) - 1 = 1$. Some applications of our main result are also presented which contains various classical results for the typical subclasses of starlike and convex functions.

1. Introduction

Let $\mathcal{A}$ denote the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Let α be a real number with $\alpha \in (0,1]$. A function $f \in \mathcal{A}$ is called *strongly starlike of order* α if it satisfies

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha$$

for all $z \in \mathbb{D}$ and *strongly convex of order* α if

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}\alpha$$

for all $z \in \mathbb{D}$. Let us denote by $\mathcal{S}^*(\alpha)$ the class of functions strongly starlike of order α, and by $\mathcal{K}(\alpha)$ the class of functions strongly convex of order α. The class $\mathcal{S}^*(\alpha)$ was introduced first by Stankiewicz [13] and by Brannan and Kirwan [2], independently. It is clear from the definitions that $\mathcal{S}^*(\alpha_1) \subset \mathcal{S}^*(\alpha_2)$ and $\mathcal{K}(\alpha_1) \subset \mathcal{K}(\alpha_2)$ for $0 < \alpha_1 < \alpha_2 \leq 1$. The case when $\alpha = 1$, i.e., $\mathcal{S}^*(1)$ and $\mathcal{K}(1)$ correspond to well known classes of starlike and convex functions respectively, and therefore all the functions which belong to $\mathcal{S}^*(\alpha)$ or $\mathcal{K}(\alpha)$ are univalent in $\mathbb{D}$. We denote by $\mathcal{S}^*$ and $\mathcal{K}$ the classes of starlike and convex functions. For the general reference of classes of starlike and convex functions, see, for instance [3].

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Mocanu [9] obtained the following result (see also [11]). Here, set

$$\rho(\alpha) = \mathrm{Tan}^{-1} \frac{\left(\dfrac{\alpha}{1-\alpha}\right)\left(\dfrac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}} \sin\left[\dfrac{\pi}{2}(1-\alpha)\right]}{1+\left(\dfrac{\alpha}{1-\alpha}\right)\left(\dfrac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}} \cos\left[\dfrac{\pi}{2}(1-\alpha)\right]} \tag{1}$$

and

$$\gamma(\alpha) = \alpha + \frac{2}{\pi}\rho(\alpha).$$

Theorem A. *$\mathcal{K}(\gamma(\alpha)) \subset \mathcal{S}^*(\alpha)$ for each $\alpha \in (0, 1]$.*

We remark that the function $\gamma(\alpha)$ is continuous and strictly increases from 0 to 1 when α moves from 0 to 1. Further investigations for the above theorem can be found in [5].

Now we shall introduce the class of functions $\mathcal{S}^*(\alpha, \beta)$ and $\mathcal{K}(\alpha, \beta)$, $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, whose members satisfy the conditions

$$\left|\arg\left\{\frac{zf'(z)}{f(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha$$

and

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha$$

for all $z \in \mathbb{D}$, respectively. We call a function $f \in \mathcal{S}^*(\alpha, \beta)$ *strongly starlike of order α and type β*. In the same way, a function $f \in \mathcal{K}(\alpha, \beta)$ is *strongly convex of order α and type β*. It is obvious that $\mathcal{S}^*(\alpha, 0) = \mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha, 0) = \mathcal{K}(\alpha)$. Also the following relations are true from the definitions;

$$\begin{array}{rl} \text{i)} & \mathcal{S}^*(\alpha_1, \beta) \subset \mathcal{S}^*(\alpha_2, \beta), \\ \text{ii)} & \mathcal{K}(\alpha_1, \beta) \subset \mathcal{K}(\alpha_2, \beta), \\ \text{iii)} & \mathcal{S}^*(\alpha, \beta_1) \supset \mathcal{S}^*(\alpha, \beta_2), \\ \text{iv)} & \mathcal{K}(\alpha, \beta_1) \supset \mathcal{K}(\alpha, \beta_2), \end{array} \tag{2}$$

for $0 < \alpha_1 < \alpha_2 \leq 1$ and $0 \leq \beta_1 < \beta_2 < 1$. That is why all functions belong to $\mathcal{S}^*(\alpha, \beta)$ or $\mathcal{K}(\alpha, \beta)$ are univalent on $\mathbb{D}$.

A sufficient condition for which $f \in \mathcal{A}$ lies in $\mathcal{S}^*(\alpha, \beta)$ was proved by the second author et al. [12]. The authors also proposed in [12] the open problem about a inclusion relationship between $\mathcal{K}(\alpha, \beta)$ and $\mathcal{S}^*(\alpha, \beta)$. However, it seems that no results concerning this question have been known.

Our main result is the following;

Theorem 1.1. *$\mathcal{K}(\gamma(\alpha), \beta) \subset \mathcal{S}^*(\alpha, \beta)$ for each $\alpha \in (0, 1]$ and $\beta \in [0, 1)$.*

The above theorem includes Theorem A as the case when $\beta = 0$.

We should notice the reader that this estimation is not sharp for each $\alpha \in (0, 1]$ and $\beta \in [0, 1)$ (see also [5]). We will discuss about this problem in section 2 with the proof of Theorem 1.1. Our main theorem yields several applications which will be shown in the last section.

2. Proof of Theorem 1.1

Our proof relies on the following lemma which was obtained by the second author [10, 11];

Lemma A. *Let $p(z)$ be analytic and satisfies $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{D}$. Let us assume that there exists a point $z_0 \in \mathbb{D}$ such that $|\arg p(z)| < \pi\alpha/2$ for $|z| < |z_0|$ and $|\arg p(z_0)| = \pi\alpha/2$ where $\alpha > 0$. Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k$$

where $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)$ when $\arg p(z_0) = \pi\alpha/2$ and $k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right)$ when $\arg p(z_0) = -\pi\alpha/2$, where $p(z_0)^{1/\alpha} = \pm ia$ and $a > 0$.

The next result will be used later;

Lemma 2.1. $\mathrm{Tan}^{-1}\alpha \geq \rho(\alpha)$ *for all $\alpha \in (0, 1]$, where ρ is defined by (1).*

Proof. Put $\phi(\alpha) = (1/(1-\alpha))\left((1-\alpha)/(1+\alpha)\right)^{\frac{1+\alpha}{2}}$. It is enough to prove that

$$\alpha \geq \frac{\alpha\phi(\alpha)\sin\left[\pi(1-\alpha)/2\right]}{1 + \alpha\phi(\alpha)\cos\left[\pi(1-\alpha)/2\right]}$$

for all $\alpha \in (0, 1]$. Since $\phi(\alpha) < 1$ because of $\phi(0) = 1$ and $\phi'(\alpha) < 0$, we obtain $\alpha > \phi(\alpha)$ and therefore

$$\frac{\alpha\sin\left[\pi(1-\alpha)/2\right]}{1 + \alpha\cos\left[\pi(1-\alpha)/2\right]} > \frac{\alpha\phi(\alpha)\sin\left[\pi(1-\alpha)/2\right]}{1 + \alpha\phi(\alpha)\cos\left[\pi(1-\alpha)/2\right]}.$$

It remains to show that

$$\alpha \geq \frac{\alpha\sin\left[\pi(1-\alpha)/2\right]}{1 + \alpha\cos\left[\pi(1-\alpha)/2\right]}$$

for all $\alpha \in (0, 1]$ and this is clear.

□

Proof of Theorem 1.1. Let us suppose that f satisfies the assumption of the theorem and let

$$p(z) = \frac{1}{1-\beta}\left(\frac{zf'(z)}{f(z)} - \beta\right).$$

Then $p(0) = 1$, and calculations show that

$$1 + \frac{zf''(z)}{f'(z)} - \beta = (1-\beta)p(z)\left\{1 + \frac{\frac{zp'(z)}{p(z)}}{(1-\beta)p(z) + \beta}\right\}. \tag{3}$$

We note that $p(z) \neq 0$ holds for all $z \in \mathbb{D}$ since $1 + zf''(z)/f'(z) - \beta \neq \infty$ on $\mathbb{D}$ from our assumption.

Now we derive a contradiction by using Lemma A. If there exists a point z_0 such that $|\arg p(z)| < \pi\alpha/2$ for $|z| < |z_0|$ and $|\arg p(z_0)| = \pi\alpha/2$, where $\alpha \in (0, 1]$, then by Lemma A, p must satisfy $z_0 p'(z_0)/p(z_0) = i\alpha k$ where $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)$ when $\arg p(z_0) = \pi\alpha/2$ and $k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right)$ when $\arg p(z_0) = -\pi\alpha/2$, where $p(z_0)^{1/\alpha} = \pm ia$ and $a > 0$.

At first we suppose that $\arg p(z_0) = \pi\alpha/2$. Then from (3) we have

$$\begin{aligned}\arg\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}-\beta\right\} &= \arg\left[(1-\beta)p(z_0)\left\{1+\frac{\frac{z_0 p'(z_0)}{p(z_0)}}{(1-\beta)p(z_0)+\beta}\right\}\right] \\ &= \frac{\pi}{2}\alpha+\arg\left\{1+\frac{i\alpha k}{(1-\beta)p(z_0)+\beta}\right\}.\end{aligned}$$

We shall estimate the second term of the second line of above. Geometric observations show that the point $1+[i\alpha k/\{(1-\beta)p(z_0)+\beta\}]$ lies on the subarc C of the circle which passes through 1, $1+i\alpha k$ and $1+[i\alpha k/p(z_0)]$, where C connects $1+i\alpha k$ and $1+[i\alpha k/p(z_0)]$ and does not pass through 1. Further, we can find out that the value $\{\arg z : z \in C\}$ attains its minimum at the end points of C. Therefore we have

$$\arg\left\{1+\frac{i\alpha k}{(1-\beta)p(z_0)+\beta}\right\} \geq \min\left\{\arg\{1+i\alpha k\},\ \arg\left\{1+\frac{i\alpha k}{p(z_0)}\right\}\right\}. \tag{4}$$

Here, the first value in the above minimum can be evaluated by $\arg\{1+i\alpha k\} \geq \mathrm{Tan}^{-1}\alpha$ since $k \geq 1$. For the second value, we note that $a^{1-\alpha}+a^{-1-\alpha}$ takes its minimum value at $a=\sqrt{(1+\alpha)/(1-\alpha)}$. Therefore

$$\begin{aligned}\arg\left\{1+\frac{i\alpha k}{p(z_0)}\right\} &= \arg\left\{1+e^{\frac{\pi}{2}(1-\alpha)i}\cdot\frac{\alpha}{2}[a^{1-\alpha}+a^{-1-\alpha}]\right\} \\ &\geq \arg\left\{1+e^{\frac{\pi}{2}(1-\alpha)i}\cdot\frac{\alpha}{2}\left[\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1-\alpha}{2}}+\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{-1-\alpha}{2}}\right]\right\} \\ &= \rho(\alpha).\end{aligned}$$

By Lemma 2.1 we conclude that

$$\arg\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}-\beta\right\} \geq \frac{\pi}{2}\alpha+\min\left\{\mathrm{Tan}^{-1}\alpha,\ \rho(\alpha)\right\} = \frac{\pi}{2}\gamma(\alpha)$$

and this contradicts our assumption.

In the same fashion, if $\arg p(z_0) = -\pi\alpha/2$ then a similar argument shows that

$$\arg\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}-\beta\right\} \leq -\frac{\pi}{2}\alpha+\max\left\{\mathrm{Tan}^{-1}(-\alpha),\ -\rho(\alpha)\right\} = -\frac{\pi}{2}\gamma(\alpha).$$

This also contradicts our assumption and our proof is completed. □

We remark that we expect this theorem to be room for improvement in our method because the inequality (4) is a rough estimation except the case when $\beta = 0$, whereas it seems to be not easy to give a precise estimation for the left hand side of (4).

3. Applications

We would like to give a further discussion to the relationship between $\mathcal{S}^*(\alpha,\beta)$ and $\mathcal{K}(\alpha,\beta)$ by using Theorem 1.1.

3.1. It is well known that a convex function is a starlike function, that is, $\mathcal{K} \subset \mathcal{S}^*$. Furthermore, Mocanu [8] showed that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$ for all $\alpha \in (0, 1]$. Now we give the next result which includes these properties as special cases;

Corollary 3.1. *$\mathcal{K}(\alpha, \beta) \subset \mathcal{S}^*(\alpha, \beta)$ for each $\alpha \in (0, 1]$ and $\beta \in [0, 1)$.*

***Proof*.** Since $\alpha \le \gamma(\alpha)$ for all $\alpha \in (0, 1]$, $\mathcal{K}(\alpha, \beta) \subset \mathcal{K}(\gamma(\alpha), \beta) \subset \mathcal{S}^*(\alpha, \beta)$ by ii) in (2) and Theorem 1.1 which is our desired inclusion. □

Corollary 3.1 yields the following property;

Corollary 3.2. *If $zf'(z) \in \mathcal{S}^*(\alpha, \beta)$, then $f \in \mathcal{S}^*(\alpha, \beta)$.*

***Proof*.** It is obvious that $g \in \mathcal{K}(\alpha, \beta)$ if and only if $zg'(z) \in \mathcal{S}^*(\alpha, \beta)$. Thus if $zg'(z) \in \mathcal{S}^*(\alpha, \beta)$ then $g \in \mathcal{K}(\alpha, \beta) \subset \mathcal{S}^*(\alpha, \beta)$ from Corollary 3.1. Hence our assertion follows if we put $f(z) = zg'(z)$. □

This corollary is equivalent to the following; *$\mathcal{S}^*(\alpha, \beta)$ is preserved by the Alexander transformation, where the Alexander transformation* [1] *is the integral transformation defined by*

$$f(z) \mapsto \int_0^z \frac{f(u)}{u} du$$

for $f \in \mathcal{A}$.

3.2. If $\alpha = 1$, then the class $\mathcal{S}^*(1, \beta)$ and $\mathcal{K}(1, \beta)$ is called *starlike of order β* and *convex of order β*, respectively. It is easy to see that $f \in \mathcal{S}^*(1, \beta)$ satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta$$

and $f \in \mathcal{K}(1, \beta)$ satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta.$$

Marx [7] and Strohhäcker [14] showed that $\mathcal{K}(1, 0) \subset \mathcal{S}^*(1, 1/2)$. Jack [4] proposed the more general problem; What is the largest number β_0 which satisfies $\mathcal{K}(1, \beta) \subset \mathcal{S}^*(1, \beta_0)$? Later MacGregor [6] and Wilken and Feng [15] answered the problem to give the exact value of β_0;

Theorem A. *$\mathcal{K}(1, \beta) \subset \mathcal{S}^*(1, \delta(\beta))$ for all $\beta \in [0, 1)$, where*

$$\delta(\beta) = \begin{cases} \dfrac{1 - 2\beta}{2^{2-2\beta}(1 - 2^{2\beta-1})}, & \text{if} \quad \beta \neq \dfrac{1}{2}, \\ \dfrac{1}{2 \log 2}, & \text{if} \quad \beta = \dfrac{1}{2}. \end{cases}$$

This estimation is sharp for each $\beta \in [0, 1)$.

Setting $\beta = 0$, we have the result of Marx and Strohhäcker. We can obtain a similar estimation to above that "$\mathcal{K}(\gamma(\alpha), \delta(\beta)) \subset \mathcal{S}^*(\alpha, \beta)$ for all $\alpha \in (0, 1]$ and $\beta \in [0, 1)$" by Theorem 1.1 since $\beta < \delta(\beta)$ for all $\beta \in [0, 1)$. However, the following problem is still open;

Open Problem. *$\mathcal{K}(\gamma(\alpha), \beta) \subset \mathcal{S}^*(\alpha, \delta(\beta))$ for each $\alpha \in (0, 1]$ and $\beta \in [0, 1)$.*

This problem implies Theorem 1.1 because $\mathcal{S}^*(\alpha, \delta(\beta)) \subset \mathcal{S}^*(\alpha, \beta)$ for all $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, and Theorem A as the case when $\alpha = 1$.

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