NORM ESTIMATE OF PRE-SCHWARZIAN DERIVATIVES OF NONLINEAR INTEGRAL TRANSFORM FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. In this note we are concerned with a nonlinear integral transform on the subclass of Bazilevič functions with the hyperbolic sup norm norm of pre-Schwarzian derivative. For our investigations we will make use of techniques of differential subordinations.

1. INTRODUCTION

Let $\mathcal{A}$ be the family of analytic functions defined in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$, and $\mathcal{ZA}$ be the subclass of $\mathcal{A}$ whose members are locally univalent in $\mathbb{D}$, namely, $\mathcal{ZA} := \{f \in \mathcal{A} : f(z)/z \neq 0,\ z \in \mathbb{D}\}$.

In 1915 Alexander [1] first observed the integral transform defined by $J[f](z) = \int_0^z f(u)/u\,du$ on the class $\mathcal{ZA}$ which maps the class of starlike functions onto the class of convex functions. Thus one might be expected that $J[f]$ always produces a univalent function for all $f \in \mathcal{S}$, where $\mathcal{S}$ is the subclass of $\mathcal{A}$ consisting of univalent functions on $\mathbb{D}$. However, in 1963 Krzyż and Lewandowski [11] gave the counter example $f(z) = z/(1-iz)^{1-i}$ which is $\pi/4$-spirallike but transformed to a non-univalent function. In 1972 Kim and Merkes [10] extended this type of transform by introducing a complex parameter $a \in \mathbb{C}$ and putting

$$J_a[f](z) := \int_0^z \left(\frac{f(u)}{u}\right)^a du$$

for $f \in \mathcal{ZA}$ where the branch is chosen so that $J_a[f](0) = 0$. In their investigation it was shown that $J_a[\mathcal{S}] \subset \mathcal{S}$ when $|a| \leq 1/4$. Otherwise it is known that if $|a| > 1/2$ then $J_a[\mathcal{S}] \not\subset \mathcal{S}$ (consider $J_a[K](z)$ and Royster's example [12] where K is the Koebe function, i.e., $K(z) = z/(1-z)^2$). Up to now, nothing better estimates have been obtained in this problem. The reader may be referred to [4] for the standard terminology in the theory of univalent functions and [6, Chapter 15] for the basic information of the integral transforms, respectively.

The purpose of this paper is to estimate the range of $|a|$ of which $J_a[f]$ is univalent in $\mathbb{D}$ when f belongs to the subclass of Bazilevič functions. Here, for $\gamma = \alpha + i\beta$ with

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$\operatorname{Re}\gamma > 0$, we say that $f \in \mathcal{A}$ is *Bazilevič of type α and β* if f satisfies

$$\operatorname{Re}\left\{ f'(z)\left(\frac{z}{f(z)}\right)^{1-\gamma}\left(\frac{z}{g(z)}\right)^{\alpha}\right\} > 0 \tag{1}$$

in $\mathbb{D}$ for some starlike functions $g \in \mathcal{A}$. In [2] it was shown that a Bazilevič function is univalent in $\mathbb{D}$. If $g(z) = z$, we denote such a subclass of Bazilevič functions by $\mathcal{U}(\gamma)$, i.e.,

$$\mathcal{U}(\gamma) := \left\{ f : \operatorname{Re} f'(z)\,(z/f(z))^{1-\gamma} > 0 \text{ for all } z \in \mathbb{D}\right\}.$$

Our main result is the following:

Theorem 1. *$J_a[\mathcal{U}(\gamma)] \subset \mathcal{S}$ if $|a| \le |\gamma/2|$.*

This theorem will be proved by making use of the estimation of the hyperbolic sup norm of the pre-Schwarzian derivatives $||T_f||_1$,

$$||T_f||_1 := \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|,$$

for functions $f \in \mathcal{U}(\gamma)$. In fact, we will have the following in Section 3:

Theorem 2. *Let $f \in \mathcal{U}(\gamma)$ and $a \in \mathbb{C}$. Then we have $||T_{J_a[f]}||_1 \le 2|a/\gamma|$.*

In the last section several problems for the norm of the pre-Schwarzian derivative and the Schwarzian derivatives concerning with the above results are considered.

2. Preliminaries

The following two theorems play central roles in our investigation. The first is an important property of differential subordinations due to Hallenbeck and Ruscheweyh. Here we recall the definition of subordination. For analytic functions f and g, we say that f *is subordinate to* g if there exists an analytic function w which maps $\mathbb{D}$ into $\mathbb{D}$ such that $w(0) = 0$ and $f(z) = g(w(z))$. This relationship is denoted by $f(z) \prec g(z)$.

Theorem A (Hallenbeck and Ruscheweyh [7])**.** *Let $p(z)$ be convex univalent in $\mathbb{D}$ with $p(0) = 1$. Let $\varphi(z)$ be analytic in $\mathbb{D}$ with $\varphi(0) = 1$ and suppose $\varphi(z) \prec p(z)$. Then for all $\gamma \ne 0$ with $\operatorname{Re}\gamma > 0$, we have $q(z) \prec p(z)$ where*

$$q(z) = \gamma z^{-\gamma}\int_0^z u^{\gamma-1}\varphi(u)du. \tag{2}$$

The second is a fundamental subordination principle in Geometric Function Theory. The original idea is due to Littlewood.

Theorem B (Kim and Sugawa [9, p.195])**.** *Let g be locally univalent in $\mathbb{D}$. For an analytic function f in $\mathbb{D}$, if $f'(\mathbb{D}) \subset g'(\mathbb{D})$, then we have $||T_f||_1 \le ||T_g||_1$. In particular, f is uniformly locally univalent on $\mathbb{D}$.*

3. Proofs of Theorem 2 and Theorem 1

Proof of Theorem 2. Let us suppose that $f \in \mathcal{U}(\gamma)$ and set $\varphi(z) := f'(z)(z/f(z))^{1-\gamma}$. Then we have $(J_\gamma[f](z))' \prec p(z) = (1+z)/(1-z)$ by Theorem A because

$$\gamma z^{-\gamma} \int_0^z u^{\gamma-1}\varphi(u)du = \left(\frac{f(z)}{z}\right)^\gamma .$$

Therefore Theorem B yields

$$\begin{aligned} ||T_{J_\gamma[f]}||_1 &\leq \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{p'(z)}{p(z)}\right| \\ &= 2. \end{aligned}$$

From the fact $||T_{J_\gamma[f]}||_1 = |\gamma| \cdot ||T_{J[f]}||_1$ we conclude that

$$||T_{J_a[f]}||_1 \leq \left|\frac{2a}{\gamma}\right| .$$

□

Proof of Theorem 1. This follows immediately from Becker's univalence criterion [3] which claims that for a function $f \in \mathcal{A}$ if $||T_f||_1 \leq 1$ then $f \in \mathcal{S}$. □

We remark that Theorem 1 and Theorem 2 are valuable when $1/2 \leq |\gamma|$ because the following theorem is known:

Theorem C (Kim and Sugawa [8, Theorem 1.1])**.** *The inequality* $||T_{J_a[f]}||_1 \leq 4|a|$ *holds for every* $f \in \mathcal{S}$ *and every complex number* a*. The bound is sharp.*

4. Further problems

4.1. **Generalization of Theorem 2.** We would like to extend the above result to more general case, for instance, when f is a Bazilevič function of type α and β. Then there exists a univalent starlike function $g(z)$ such that the inequality (1) holds. We know that if $g(z) = zh'(z)$ then h is convex, so that in view of this (1) is equivalent to

$$\operatorname{Re}\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1-\gamma}\left(\frac{1}{h'(z)}\right)^{\alpha}\right\} > 0.$$

Here we shall propose the following problem:

Question 3. *Let* φ *be analytic in* $\mathbb{D}$ *with* $\varphi(0) = 1$ *and* $h \in \mathcal{A}$ *be convex. Let* γ *be a complex constant with* $\operatorname{Re}\gamma > 0$ *and* $\alpha = \operatorname{Re}\gamma$*. If*

$$\varphi(z)h'(z)^{-\alpha} \prec p(z) = \frac{1+z}{1-z},$$

then

$$\Phi(z)h'(z)^{\alpha} \prec p(z)$$

or

$$\Phi(z)h'(z)^{-\alpha} \prec p(z)$$

holds, where Φ *is the function defined by* (2)*.*

We remark that in the above question if we choose $h(z) = z$ then it is nothing but the theorem of Hallenbeck and Ruscheweyh.

If the above question is true, then for a function f which is Bazilevič of type α and β it follows from Theorem B that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{J_\gamma[f]''}{J_\gamma[f]'}-\alpha\frac{h''(z)}{h'(z)}\right| \le \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{p'(z)}{p(z)}\right|$$

and hence

$$||T_{J_\gamma[f]}||_1 \le 2+4\alpha.$$

Here we have used a result of Yamashita [13] which states that if $h \in \mathcal{A}$ is convex then $||T_h||_1 \le 4$. Therefore we will obtain a generalization of Theorem 2.

4.2. **Theorem B and the Schwarzian derivatives.** For an analytic function f with $f' \neq 0$, the Schwarzian derivative S_f and the hyperbolic sup norm of S_f are defined by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

and

$$||S_f||_2 := \sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_f|,$$

respectively.

Theorem B has a wide range of applications so that we might hope that the inequality $||S_f||_2 \le ||S_g||_2$ also holds for functions f, g with $f'(\mathbb{D}) \subset g'(\mathbb{D})$. However, it can be shown that the inequality does not always hold under this situation:

Theorem 4. *Let g be locally univalent and f be analytic in D respectively. If $f'(\mathbb{D}) \subset g'(\mathbb{D})$, then we have*

$$||S_f||_2 \le ||S_g||_2 + ||T_w||_1 \cdot ||T_g||_1,$$

where $w = g^{-1} \circ f$. In particular f is uniformly locally univalent on $\mathbb{D}$.

Proof. It concludes easily with the same technique as Theorem B. □

The proof of Theorem B, namely Theorem 4 also, relies on the Schwarz-Pick lemma so that it seems not to be obtained better estimation than the above in this direction. We may use Theorem 4 to derive a norm estimate for $||S_{J_\gamma[f]}||_2$ with the theorem due to Duren, Shapiro and Shields [5] which claims $||S_f||_2 \le 4||T_f||_1 + \frac{1}{2}(||T_f||_1)^2$ for $f \in \mathcal{A}$. In any case, we have to handle the term $||T_w||_1$ to make use of Theorem 4.

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